



2022 CMWMC Guts Round Solutions

1. Assume the speed of sound is 343 m/s. Anastasia and Bananastasia are standing in a field in front of you. When they both yell at the same time, you hear Anastasia's yell 5 seconds before Bananastasia's yell. If Bananastasia yells first, and then Anastasia yells when she hears Bananastasia yell, you hear Anastasia's yell 5 seconds after Bananastasia's yell. What is the distance between Anastasia and Bananastasia in meters?

Proposed by David Tang

Answer. 3430

Solution. If Bananastasia's yell takes x seconds to reach you, Anastasia's yell takes $x - 5$ seconds. Thus, we must delay Anastasia's yell by 10 seconds to hear her yell 5 seconds after Bananastasia's yell. Therefore, the distance between them is 10 seconds times the speed of sound which is 343 m/s, so the final answer is 3430.

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2. Michelle picks a five digit number with distinct digits. She then reverses the digits of her number and adds that to her original number. What is the largest possible sum she can get?

Proposed by Connor Gordon

Answer. 184147

Solution. Letting the number be $abcde$, adding the number and its reversal gives $10001(a + e) + 1010(b + d) + 200c$. We thus want a and e to be as large as possible, so 9 and 8, b and d to be the next largest possible, so 7 and 6, and c to be the next largest possible, so 5. This yields an answer of $10001(9 + 8) + 1010(7 + 6) + 200(5) = 184147$.

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3. Twain is trying to crack a 4-digit number combination lock. They know that the second digit must be even, the third must be odd, and the fourth must be different from the previous three. If it takes Twain 10 seconds to enter a combination, how many hours would it take them to try every possible combination that satisfies these rules?

Proposed by Ayush Mediratta

Answer. 5

Solution. There are 5 evens and 5 odds for the second, third digits. Now, we consider two cases: if the first digit equals the second or third, or if it is different. In the former case, there are 8 options for the first digit, and the last digit has 7 options. In the later case, there are 2 options for the first digit and 8 options for the last digit. The total number of combinations are then

$$5 \cdot 5 \cdot (8 \cdot 7 + 2 \cdot 8) = 1400 + 400 = 1800.$$

The total time it takes in hours is $1800 \cdot 10 / 3600 = 5$.



4. $\triangle ABC$ is an isosceles triangle with $AB = BC$. Additionally, there is D on BC with $AC = DA = BD = 1$. Find the perimeter of $\triangle ABC$.

Proposed by Brandon Dong

Answer. $2 + \sqrt{5}$

Solution. Notice that $\angle DCA = \angle ACB$, so the base angles of two isosceles triangle $\triangle DAC$, $\triangle ABC$ are same, hence they are similar. Then letting y be the length of $AB = BC$ we have that

$$\frac{AB}{AC} = \frac{y}{1} = \frac{DA}{DC} = \frac{1}{y-1}.$$

Solving for y we have that $y = (1 + \sqrt{5})/2$ and the perimeter is $2y + 1$ for a total of $2 + \sqrt{5}$.

5. Let r be the positive solution to the equation $100r^2 + 2r - 1 = 0$. For an appropriate A , the infinite series $Ar + Ar^2 + Ar^3 + Ar^4 + \dots$ has sum 1. Find A .

Proposed by Lucas Liu

Answer. $\sqrt{101}$

Solution. Note that $100r^2 + 2r - 1 = 0$ implies that $(1 - r)^2 = 101r^2$, or $(1 - r) = \sqrt{101} \cdot r$. By the formula for the infinite geometric series sum, we get that

$$Ar + Ar^2 + Ar^3 + Ar^4 + \dots = A \cdot \frac{r}{1 - r} = \frac{A}{\sqrt{101}} = 1.$$

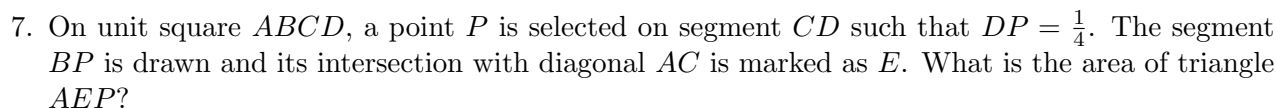
where we have used the above relation. It follows that $A = \sqrt{101}$.

6. Let $N(k)$ denote the number of real solutions to the equation $x^4 - x^2 = k$. As k ranges from $-\infty$ to ∞ , the value of $N(k)$ changes only a finite number of times. Write the sequence of values of $N(k)$ as an ordered tuple (i.e. if $N(k)$ went from 1 to 3 to 2, you would write $(1, 3, 2)$).

Proposed by Connor Gordon

Answer. $(0, 2, 4, 3, 2)$

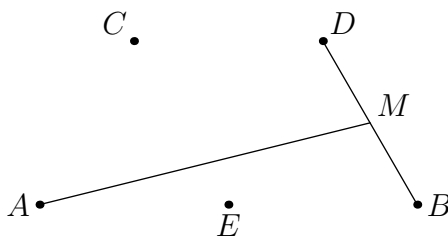
Solution. Considering the graph of $y = x^2 - x^4$, we can see that $x = 0$ gives $y = 0$, $0 < x < 1$ gives $y < 0$, and $x \geq 1$ gives $y \geq 0$ (and mirrored on the negative end). Clearly $x^4 - x^2$ is bounded below by, say, 1 (given that it's only negative for $|x| < 1$ in which the negative part is less than 1), so initially $N(k)$ is 0 for large negative k , then jumps to 2 momentarily when k hits the minimum value on $(0, 1)$, then jumps to 4 between this minimum value and 0, then jumps to 3 momentarily at $k = 0$, and finally stays at 2 for all $k > 0$. This yields the sequence $(0, 2, 4, 3, 2)$.



8. Five distinct points are arranged on a plane, creating ten pairs of distinct points. Seven pairs of points are distance 1 apart, two pairs of points are distance $\sqrt{3}$ apart, and one pair of points is distance 2 apart. Draw a line segment from one of these points to the midpoint of a pair of these points. What is the longest this line segment can be?

Note that in the diagram, we picked a point to be point C . We must pick three points for our set, so one of them must be off of the line segment $\overline{AB}$. We can pick any one of the four remaining points to be point C , and by symmetry, it doesn't matter which one in particular we pick.

We can't pick P_2 because that would make $BC = CP_2 = P_2B = \sqrt{3}$, which has too many lengths of $\sqrt{3}$. We can't pick P_3 because that would make $AB = CP_3 = 2$, creating too many lengths of 2. Therefore we must pick $D = P_1$ and $E = P_0$, which we confirm to make the correct lengths.



The longest segment we can draw is from A to midpoint M of $\overline{BD}$, as shown. (Equivalently, we can draw a segment from B to the midpoint of $\overline{AC}$.) If our segment starts at E , then our segment must be shorter than 1, and if our segment starts at C or D , then our segment must be shorter than $\sqrt{3}$.

Note that $BD = 1$, $AD = \sqrt{3}$, and $AB = 2$, so $\angle ADB = 90^\circ$. By the Pythagorean theorem,

$$AM = \sqrt{\sqrt{3}^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{13}}{2}.$$

9. The inhabitants of Mars use a base 8 system. Mandrew Mellon is competing in the annual Martian College Interesting Competition of Math (MCICM). The first question asks to compute the product of the base 8 numerals 1245415_8 , 7563265_8 , and 6321473_8 . Mandrew correctly computed the product in his scratch work, but when he looked back he realized he smudged the middle digit. He knows that the product is $1014133027\blacksquare2766204113_8$. What is the missing digit?

Proposed by Alan Abraham

Answer. 4

Solution. Since in base 8 modulo 7 is equivalent to summing the digits we know that

$$1245415_8 \equiv 1 + 2 + 4 + 5 + 4 + 1 + 5 \equiv 1 \pmod{7}$$

$$7563265_8 \equiv 7 + 5 + 6 + 3 + 2 + 6 + 5 \equiv 6 \pmod{7}$$

$$6321473_8 \equiv 6 + 3 + 2 + 1 + 4 + 7 + 3 \equiv 5 \pmod{7}$$

If x is the smudged digit, then we know that our hidden product is

$$\begin{aligned} &1 + 0 + 1 + 4 + 1 + 3 + 3 + 0 + 2 + 7 + x \\ &+ 2 + 7 + 6 + 6 + 2 + 0 + 4 + 1 + 1 + 3 \equiv 5 + x \pmod{7}. \end{aligned}$$

This means that we must have that

$$1 \cdot 6 \cdot 5 \equiv 5 + x \pmod{7} \implies 4 \equiv x \pmod{7}$$

So the only possible digit for x is $\boxed{4}$

10. Eve has nine letter tiles: three C's, three M's, and three W's. If she arranges them in a random order, what is the probability that the string "CMWMC" appears somewhere in the arrangement?

Proposed by Connor Gordon

Answer. $1/28$

Solution. Temporarily treating all of the letters as distinct, there are $3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 108$ ways to choose which letters form the "CMWMC." There are then $5!$ ways to arrange the "CMWMC" (treated as a block) and the other four letters. There are $9!$ total arrangements, for a probability of $108 \cdot 5! / 9! = 1/28$.

11. Bethany's Batteries sells two kinds of batteries: C batteries for \$4 per package, and D batteries for \$7 per package. After a busy day, Bethany looks at her ledger and sees that every customer that day spent exactly \$2021, and no two of them purchased the same quantities of both types of battery. Bethany also notes that if any other customer had come, at least one of these two conditions would've had to fail. How many packages of batteries did Bethany sell?

Proposed by Jacob Weiner

Answer. 28548

Solution. One customer could have bought 500 packages of C batteries and 3 packages of D batteries. Then you can always get another customer by subtracting 7 packages of C and adding 4 packages of D, until $C < 7$. $\lfloor \frac{500}{7} \rfloor = 71$, so you can do this 71 times, reaching 3 packages of C and 287 packages of D. So the total is 72 customers.

Customer 0 buys 503 packages, Customer 1 buys $503 - 3$ packages, etc., and Customer n buys $503 - 3n$ packages. So the total is

$$\sum_{n=0}^{71} 503 - 3n = 503 \cdot 72 - 3 \cdot \frac{71 \cdot 72}{2} = 28548$$

packages.

12. A deck of cards consists of 30 cards labeled with the integers 1 to 30, inclusive. The cards numbered 1 through 15 are purple, and the cards numbered 16 through 30 are green. Lilith has an expansion pack to the deck that contains six indistinguishable copies of a green card labeled with the number 32. Lilith wants to pick from the expanded deck a hand of two cards such that at least one card is green. Find the number of distinguishable hands Lilith can make with this deck.

Proposed by Justin Hsieh

Answer. 361

Solution. There are 31 distinguishable cards in the expanded deck. There are $\frac{1}{2} \cdot 31 \cdot 30 = 465$ ways to pick a hand of 2 distinguishable cards from the deck. Of these hands, $\frac{1}{2} \cdot 15 \cdot 14 = 105$ of

these consist entirely of purple cards, so $465 - 105 = 360$ hands have at least one green card. We must also add the hand that consists of two cards numbered 32, since the deck contains multiple copies of this card. There are $360 + 1 = 361$ total hands that can now be made.

13. An equiangular 12-gon has side lengths that alternate between 2 and $\sqrt{3}$. Find the area of the circumscribed circle of this 12-gon.

Proposed by Connor Gordon

Answer. 13π

Solution. Extend the sides of length 2 to form a regular hexagon of side length 4. Then drop a perpendicular of length $2\sqrt{3}$ from the center and use the Pythagorean theorem to get the circumradius as

$$\sqrt{(2\sqrt{3})^2 + 1^2} = \sqrt{13}.$$

The answer follows.

14. For positive integers n , let $\sigma(n)$ denote the number of positive integer factors of n . Then $\sigma(17280) = \sigma(2^7 \cdot 3^3 \cdot 5) = 64$. Let S be the set of factors k of 17280 such that $\sigma(k) = 32$. If p is the product of the elements of S , find $\sigma(p)$.

Proposed by Justin Hsieh

Answer. 432

Solution. The number of factors of a positive integer $p^a q^b r^c$ is equal to $(a+1)(b+1)(c+1)$, if p, q, r are prime. In order for $\sigma(p^a q^b r^c)$ to be a power of 2, all of a, b, c must be one less than a power of two. In particular, in order for a divisor $2^a 3^b 5^c$ of 17280 to have 32 factors, it must be that exactly one of $a+1, b+1, c+1$ is half of its original value (as $2^a 3^b 5^c = 2^7 3^3 5^1$), and the other two remain the same. The elements of S are $2^3 \cdot 3^3 \cdot 5$, $2^7 \cdot 3 \cdot 5$, and $2^7 \cdot 3^3$. Their product is $2^{17} \cdot 3^7 \cdot 5^2$, which has $18 \cdot 8 \cdot 3 = 432$ factors.

15. How many odd 3-digit numbers have exactly four 1's in their binary (base 2) representation? For example, $225_{10} = 11100001_2$ would be valid.

Proposed by Ethan Gu

Answer. 67

Solution. First notice since we have odd numbers, the smallest binary digit must be a one. Now case on possibilities for the highest one, which can have order $2^9, 2^8, 2^7$, or 2^6 . The other two ones can be placed anywhere else except for in the 2^6 case, where we need a one in the 2^5 place, and can't have a one in the 2^1 place. This gives us

$$\binom{8}{2} + \binom{7}{2} + \binom{6}{2} + 3 = 67.$$

16. Let x and y be non-negative integers. We say point (x, y) is *square* if $x^2 + y$ is a perfect square. Find the sum of the coordinates of all distinct *square* points which also satisfy $x^2 + y \leq 64$.

Proposed by Ethan Gu

Answer. 1080

Solution. Notice there are $n + 1$ square points for a perfect square n^2 . For example, square points summing to 3^2 are:

$$(0, 9), (1, 8), (2, 5), (3, 0)$$

The sum of coordinates of these points is close to $(n + 1)(n^2)$ but this is an overestimate as square points with $x > 1$ have a sum of coordinates $< n^2$. We just need to subtract the difference, which will be $x^2 - x$ for each square point with $x > 1$. Each difference will be counted $n - x + 1$ times when finding the total sum of coordinates for the squares from 1 to 64. We now have the sum

$$\begin{aligned} & \sum_{i=1}^8 (i + 1)(i^2) - \sum_{i=1}^8 (8 - i)((i + 1)^2 - (i + 1)) \\ &= \sum_{i=1}^8 (i^3 + i^2) - \sum_{i=1}^8 (8(i^2 + i) - (i^3 + i^2)) \\ &= \sum_{i=1}^8 2i^3 - 6i^2 - 8i = 1080. \end{aligned}$$

17. Two integers a and b are randomly chosen from the set $\{1, 2, 13, 17, 19, 87, 115, 121\}$, with $a > b$. What is the expected value of the number of factors of ab ?

Proposed by Ethan Gu

Answer. $171/28$

Solution. We find expected value by finding the total number of factors and dividing by the total number of distinct pairs of (a, b) . Fix b and consider all values of a for each b . This casework can be sped up by noticing that each element is co-prime to every other element in the set, so the number of factors of ab is just equal to multiplying the amount of factors of a with the amount of factors of b . The total sum of factors is 171, dividing by the total number of pairs 28, we obtain $171/28$.

18. Marnie the Magical Cello is jumping on nonnegative integers on number line. She starts at 0 and jumps following two specific rules. For each jump she can either jump forward by 1 or jump to the next multiple of 4 (the next multiple must be strictly greater than the number she is currently on). How many ways are there for her to jump to 2022? (Two ways are considered distinct only if the sequence of numbers she lands on is different.)

Proposed by Alan Abraham

Answer. 4^{505}

Solution. Because Marnie can only jump to multiples of 4 or jump forward by 1, she will always hit every multiple of 4. If she is currently on some multiple of 4, say $4n$, then there are 4 ways to get to $4(n+1)$. So to get from 0 to $4n$, there are 4 ways to get to each of the n multiples of 4, so there are a total of 4^n different sequences of jumps. Hence, she can get to 2020 in 4^{505} different ways. From 2020 there is only 1 way to get to 2022 by jumping forward twice. Hence, the total number of sequences of jumps is 4^{505} .

19. The polynomial $x^4 + ax^3 + bx^2 - 32x$, where a and b are real numbers, has roots that form a square in the complex plane. Compute the area of this square.

Proposed by Connor Gordon

Answer. 8

Solution. Note that 0 is a root of the polynomial. Since the coefficients are real, the roots must come in conjugate pairs, so in particular the polynomial has another real root r which must be the opposite vertex of the square from 0. We can then compute the other vertices must be $r/2 \cdot (1 \pm i)$.

By Vieta's the product of the roots is $r^3/2 = 32 \Rightarrow r = 4$. The square thus has diagonal length 4 and thus area 8.

20. Tetrahedron $ABCD$ has equilateral triangle base ABC and apex D such that the altitude from D to ABC intersects the midpoint of $\overline{BC}$. Let M be the midpoint of $\overline{AC}$. If the measure of $\angle DBA$ is 67° , find the measure of $\angle MDC$ in degrees.

Proposed by Justin Hsieh

Answer. 46

Solution. Tetrahedron $ABCD$ has a plane of reflectional symmetry about plane ADL , where L is the midpoint of $\overline{BC}$. This means $\angle DBA = \angle DCA = 67^\circ$. Since M is on $\overline{AC}$, $\angle DCA = \angle DCM = 67^\circ$. Also, $LM = LC$ since $\triangle LCM \sim \triangle BCA$ in the ratio 1 : 2 by SAS similarity, and since $\triangle ABC$ is equilateral. Therefore $DM = DC$, since $\triangle DLC \cong \triangle DLM$ by SAS congruence. (If P is any point on the plane containing ABC , then the measure of $\angle DLP$ is 90° .) Therefore $\triangle CDM$ is isosceles with $DM = DC$, so $\angle MDC = 180^\circ - 2\angle DBC = 180^\circ - 2 \cdot 67^\circ = 46^\circ$.

21. Last year's high school graduates started high school in year $n - 4 = 2017$, a prime year. They graduated high school and started college in year $n = 2021$, a product of two consecutive primes. They will graduate college in year $n + 4 = 2025$, a square number. Find the sum of all $n < 2021$ for which these three properties hold. That is, find the sum of those $n < 2021$ such that $n - 4$ is prime, n is a product of two consecutive primes, and $n + 4$ is a square.

Proposed by Jacob Weiner

Answer. 514

Solution. Let $n + 4 = s^2$. Then $n = (s + 2)(s - 2)$. So $s + 2, s - 2$ must be consecutive primes which differ by four. (Or, $s - 2 = 1$, but then $n = 5$ which doesn't count.)

We are only considering $s < 45$ because $45^2 = 2025$. The primes less than 45 are:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43.

The s values which induce a pair which differs by 4 are: $s = 9, 15, 21, 39$.

Now we need to check if $s^2 - 8$ is prime for each of those: $s = 9 \rightarrow s^2 - 8 = 73$ is prime
 $s = 15 \rightarrow s^2 - 8 = 217$ is divisible by 7 easily. $s = 21 \rightarrow s^2 - 8 = 433$ is prime (only requires trial division by primes 11-19 if smart) $s = 39 \rightarrow s^2 - 8 = 1513 = 17 \cdot 89$ is divisible by 17.

So there are two such s , so there are two such n , namely $n = 77, 437$. Adding them, 514.

22. For monic quadratic polynomials $P = x^2 + ax + b$ and $Q = x^2 + cx + d$, where $1 \leq a, b, c, d \leq 10$ are integers, we say that P and Q are *friends* if there exists an integer $1 \leq n \leq 10$ such that $P(n) = Q(n)$. Find the total number of ordered pairs (P, Q) of such quadratic polynomials that are friends.

Proposed by Kyle Lee

Answer. 1604

Solution. We have that $P(n) = Q(n)$ if and only if $(a - c)n = d - b$. If $a - c = 0$ and $d - b = 0$, it is easy to see that P and Q are friends (they are equal), so there are $(10)^2 = 100$ such pairs.

Now, note that if $(a - c) \mid (d - b)$, then we will have a valid (P, Q) as long as either $a > c, d > b$ or $a < c, d < b$. This is because $\max(|(d - b)|, |(a - c)|) \leq 9$, so n will definitely be restricted in the valid range.

Let S_n be the number of ordered pairs (j, k) , where $1 \leq j, k \leq 10$ are integers such that $|j - k| = n$. It is easy to see that $S_n = 10 - n$ for all integers $1 \leq n \leq 10$. Then, we have

$$2 \cdot (9(9 + 8 + \cdots + 1) + 8(8 + 6 + 4 + 2) + 7(7 + 4 + 1) + 6(6 + 2) + 5^2 + 4^2 + 3^2 + 2^2 + 1^2) = 1504$$

such pairs in this case.

The final total is $100 + 1504 = \boxed{1604}$

23. A three-dimensional solid has six vertices and eight faces. Two of these faces are parallel equilateral triangles with side length 1, $\triangle A_1A_2A_3$ and $\triangle B_1B_2B_3$. The other six faces are isosceles right triangles— $\triangle A_1B_2A_3$, $\triangle A_2B_3A_1$, $\triangle A_3B_1A_2$, $\triangle B_1A_2B_3$, $\triangle B_2A_3B_1$, $\triangle B_3A_1B_2$ —each with a right angle at the second vertex listed (so for instance $\triangle A_1B_2A_3$ has a right angle at B_2). Find the volume of this solid.

Proposed by Alan Abraham

Answer. $1/3\sqrt{2}$

Solution. Note that this figure is a part of a cube where A_iB_i are spatially opposite vertices. Denote A_0 and B_0 as the missing vertices of the cube so that $A_0A_1A_2A_3$ and $B_0B_1B_2B_3$ are right tetrahedra. The entire cube has sidelength $1/\sqrt{2}$, so its volume is $1/2\sqrt{2}$. The regions that have been removed from the cube to make the figure are the two tetrahedra $A_0A_1A_2A_3$ and $B_0B_1B_2B_3$. These in total make up $1/3$ of the volume of the cube, so the figure has volume $1/3\sqrt{2}$.

24. The digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are each colored red, blue, or green. Find the number of colorings such that any integer $n \geq 2$ has that
- (a) If n is prime, then at least one digit of n is not blue.
 - (b) If n is composite, then at least one digit of n is not green.

Proposed by Oliver Hayman

Answer. 30

Solution. Note that 111, 22, 33, 44, 55, 66, 77, 88, and 99 are all composite, so all digits except possibly 0 are not green. This is sufficient for the second condition, as all positive integers contain a nonzero digit. Additionally, 11, 2, 3, 5, and 7 are all primes, so 1, 2, 3, 5, 7 all must be red. Now we need to color the digits 0, 4, 6, 8, 9. Note that all primes containing only these digits must end in 9, as any such prime must be odd.

If 9 is red, since all primes must contain 9 or any of the other red digits, 0 can be red, blue, or green, 4 can be red or blue, 6 can be red or blue, and 8 can be red or blue. This gives $2 \cdot 2 \cdot 2 \cdot 3 = 24$ colorings.

If 9 is blue, note that 449 and 89 are both primes, so 4 and 8 must be red. Additionally, note that any number containing only the digits 0, 6, 9 is divisible by 3 and must be composite. Therefore, it is sufficient to color 4 and 8 red, so 0 can be red, green, or blue, and 6 can be red or blue, giving $2 \cdot 3 = 6$ colorings.

Therefore, the total number of colorings is 30.